

Metric Spaces and Their Geometries via *Excel*

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Abstract: *This paper provides a fresh approach for an introductory exploration of metric spaces, illustrating methods for using Excel creatively to discover interesting insights into mathematics. We use diverse examples of metrics to provide insights into the concept of distance, and see how it arises in the areas of geometry, calculus, and other well-known areas of mathematics. Along the way, we encounter inequalities and other aspects of classical mathematics, provide new problem solving and teaching techniques, and present a means of discovering patterns by employing spreadsheets to enhance the visual interpretation of a vital field of mathematics.*

1. Introduction

For many years numerous people have contributed to the development of a wide range of creative ways to employ spreadsheets effectively in the teaching of pure and applied mathematics. A new opportunity for extending the scope of these endeavors arose in 2013 in developing and teaching an undergraduate on-line class in metric spaces for Divine Word University in Madang, Papua New Guinea.

This paper describes some of those developments. Because many of our models also involve concepts from areas such as calculus, analysis, inequalities, geometry, and linear algebra, they not only provide tools for the study and teaching of metric spaces, but they also furnish new insights into other subjects in the mathematics and computing science curricula. We supply the underlying *Excel 2010* files with annotations for those interested [11]-[23].

The use of a spreadsheet such as *Microsoft Excel* in our endeavors enables us to design mathematical models that closely follow standard ways of presenting and learning mathematics using software that is readily available and familiar to students. Moreover, it provides students with new and creative skills in applying a fundamental and highly-valued tool of the workplace.

For this specific class, it was helpful to be able to create interesting, animated spreadsheet models of topics that could be used in live Internet presentations, and to largely avoid a heavy reliance on the traditional definition-theorem-proof teaching format. The use of *Excel* helped students in developing problem solving skills, especially in discovering patterns for the geometry of different metrics. It provided students with an accessible tool for examining various metrics, constructing geometric objects, and initiating new approaches and insights for their studying. During the term, students could access *Excel* models and extensive class notes through *Moodle*.

The examples provide here can be used for at least three important purposes:

- The examples provide teachers with interactive illustrations for classroom presentations.
- The techniques form an effective means for students to discover patterns.
- The approach presents a sound basis for individual and group student projects that integrate mathematical concepts and spreadsheet implementation skills.

2. Foundation Information

Definition: A metric space consists of a set X together with a distance function, $d : X \times X \rightarrow \mathbb{R}$, where \mathbb{R} is the space of real numbers, having the following three properties:

- i) $d(a, b) \geq 0$ for all $a, b \in X$, with equality if and only if $a = b$
- ii) $d(a, b) = d(b, a)$ for all $a, b \in X$
- iii) $d(a, b) \leq d(a, c) + d(c, b)$ for all $a, b, c \in X$ (triangle inequality)

We see these properties for distance when we look at familiar road maps. For example, Figure 2.1 provides a stylized map and distance table for roads and towns near Martin, Tennessee, USA. Properties (i) and (ii) are readily apparent from the table. We can examine property (iii) either manually or through an interactive *Excel* model such as the one at the right. Here we observe that a smaller example for classroom illustration is easier to implement if all cities can be connected by straight lines. To do this, we include a virtual route from Martin to Paris, although actual travel goes through Dresden. We also incorporate a slightly more involved version among our *Excel* files.

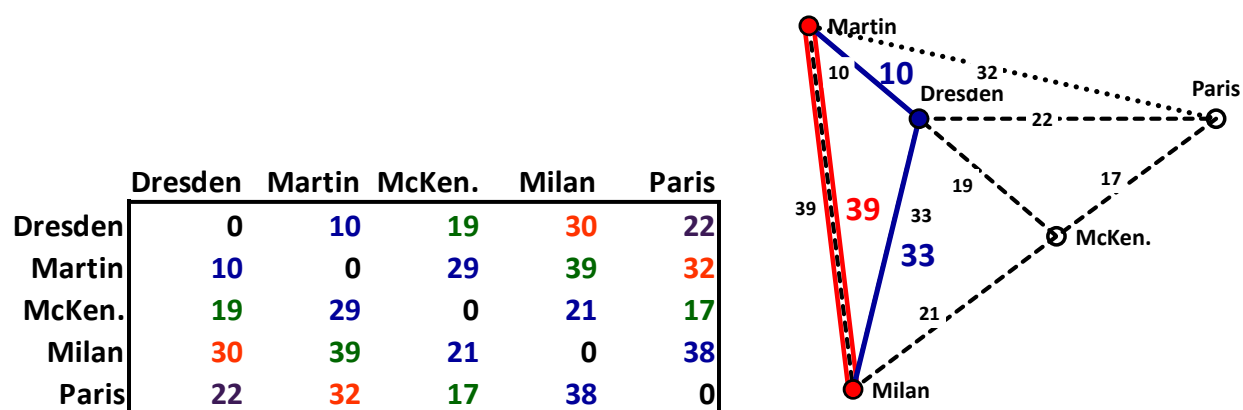


Figure 2.1. Road Map Metric

The most familiar examples of metric spaces are the real numbers \mathbb{R} with distance function $d(a, b) = |b - a|$, the real Euclidean plane \mathbb{R}^2 with distance function $d((x_1, y_1), (x_2, y_2)) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$, and n -dimensional Euclidean space \mathbb{R}^n with a similar distance function.

However, there many more interesting metric spaces for students to explore in which they will encounter new mathematical concepts and techniques, and for educators to design useful teaching visualizations. In this paper we present a few of these that we implement using *Microsoft Excel 2010*. We will not prove that the metrics listed satisfy the conditions for a metric, but refer readers to other sources [7], [8] for these results.

3. Examples of Metrics Spaces

3.1. Euclidean \mathbb{R}^2 Metric: There are many ways to use *Excel* to create Euclidean metric objects in \mathbb{R}^2 . For example, in Figure 3.1a we create an xy -graph of a circle by using the parametric equations

$x = x_0 + r \cos t, y = y_0 + r \sin t, 0 \leq t \leq 2\pi$, thus locating the set of points whose distance from the point (x_0, y_0) is r . Although *Excel* does not provide a direct way to fill in the interior of a closed curve, we do this by drawing a series of line segments, as indicated in Figure 3.1b. To fill in the area, we increase the number of segments and the width of the lines. By adjusting the line style for the boundary, in Figure 3.1c we produce the open unit disk (or ball) centered at $(a, b) = (1.4, 1.2)$. We show an outline of our *Excel* worksheet and its equations in Figure 3.2. A full description of the methods used appears in [6] and in our annotated *Excel* files.

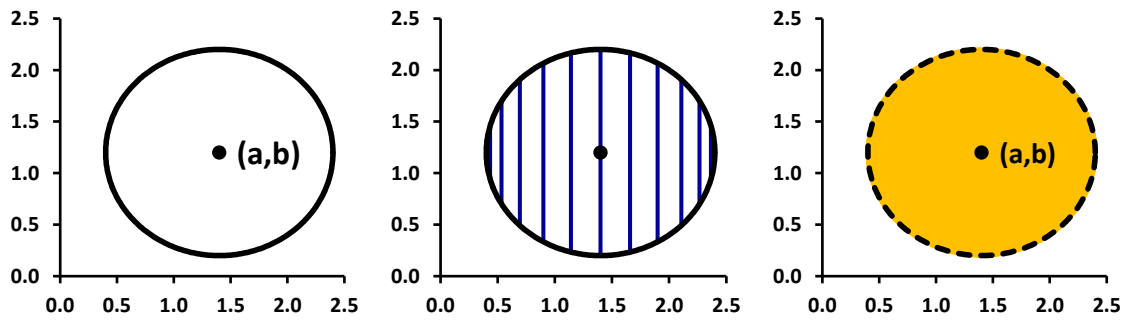


Figure 3.1 Unit Circle and Open Unit Disk in \mathbb{R}^2 .

	A	B	C	D	E	F	G	H	I
3		radius:	1	1					
4		center:	1.4	1.2					
5			circle			interior fill		center	
6	deg	t rad	x	y	n	x	y	x	y
7	0	0	2.4	1.2	0	2.4	1.2	1.4	1.2
8	1	0.017	2.4	1.217	360	2.4	1.2		
9	2	0.035	2.399	1.235					
10	3	0.052	2.399	1.252	1	2.4	1.217		
11	4	0.07	2.398	1.27	359	2.4	1.183		
12	5	0.087	2.396	1.287					

	A	B	C	D
3		radius:	1	=C3
4		center:	1.4	1.2
5			circle	
6	deg	t rad	x	y
7	0	=RADIANS(A7)	=C\$3*COS(\$B7)+C\$4	=D\$3*SIN(\$B7)+D\$4
8	=1+A7	=RADIANS(A8)	=C\$3*COS(\$B8)+C\$4	=D\$3*SIN(\$B8)+D\$4
9	=1+A8	=RADIANS(A9)	=C\$3*COS(\$B9)+C\$4	=D\$3*SIN(\$B9)+D\$4
10	=1+A9	=RADIANS(A10)	=C\$3*COS(\$B10)+C\$4	=D\$3*SIN(\$B10)+D\$4
11	=1+A10	=RADIANS(A11)	=C\$3*COS(\$B11)+C\$4	=D\$3*SIN(\$B11)+D\$4
12	=1+A11	=RADIANS(A12)	=C\$3*COS(\$B12)+C\$4	=D\$3*SIN(\$B12)+D\$4

Figure 3.2 *Excel* Output and Formulas for Unit Circle and Disk

As an illustrative example, suppose that we wish to find all points in \mathbb{R}^2 that are equidistant from two given points. For this example we first use algebra to find the points of intersection of the circles of radius r whose centers are the two given points. We then use those points together with our techniques for generating circles to form the diagram of Figure 3.3. If we then link the cell containing the radius to a scroll bar, we animate the creation process for drawing the perpendicular bisector of the segment between the two points as we increase r . Later we use this idea for a similar construction with another metric. Additional related examples can be found in [2], [6].

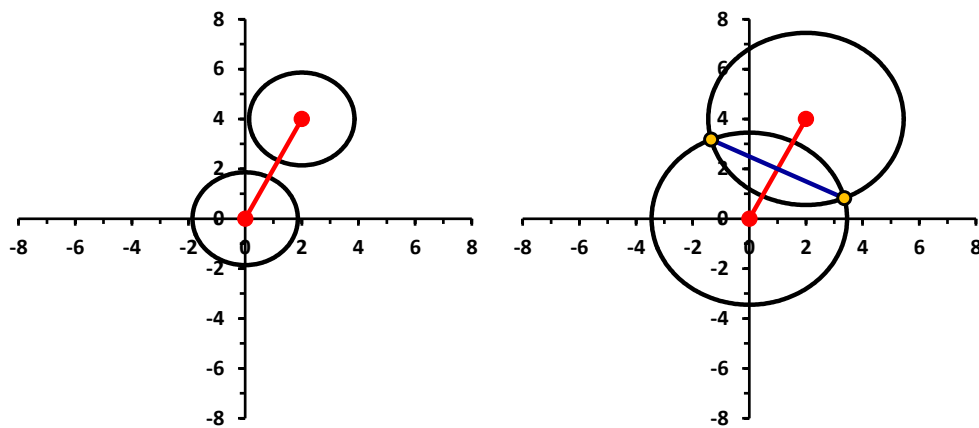


Figure 3.3 Locus of Points Equidistant from Two Given Points (Euclidean)

3.2. Taxicab Metric: Let X be the real plane, \mathbb{R}^2 , but now use the taxicab distance defined by $d((x_1, y_1), (x_2, y_2)) = |x_2 - x_1| + |y_2 - y_1|$. The name taxicab comes from the fact that this measure of distance is analogous to that used for taxicabs as they traverse the streets of a city having a rectangular street grid, where one can travel only in horizontal or vertical directions. Thus, in Figure 3.4a the distance between the locations $a = (-3, 2)$ and $b = (4, -4)$ is $d(a, b) = |4 - (-3)| + |-4 - 2| = 13$. In Figure 3.4a we also see that a trip of that length can be made in a variety of ways, with the intermediate corners interpreted as points lying between a and b .

However, in this metric space we are not limited to traveling on lines with integer coordinates. Thus, in Figure 3.4b we see that the distance between $a = (-2.5, 3.3)$ and $b = (4.3, -2.6)$ is $d(a, b) = 12.7$.

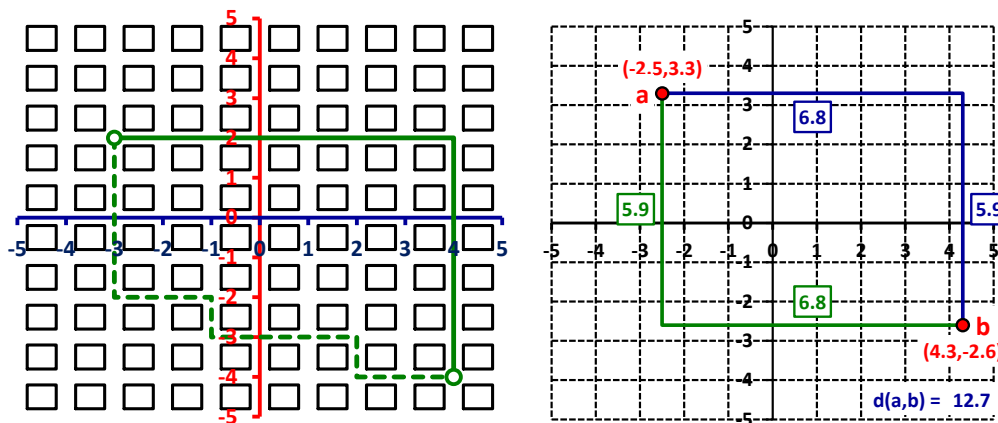


Figure 3.4. Taxicab Distance

In working with this metric by hand, it is convenient to use graph paper. A spreadsheet provides us with a similar medium through which we can gain insights before creating more advanced illustrations. In Figure 3.5a we show an xy -scatter chart produced from the *Excel* layout in Figure 3.5b to find the points that are distance $r = 4$ from point $(a, b) = (0, 0)$. Columns C:D are used to generate a grid with gaps of size 0.5. We start by first entering $C4 = D4 = -5$, and then the formulas

C5: =IF(D4=5,0.5+C4,C4), D5: =IF(D4=5,-5,0.5+D4), and afterwards filling down the two columns. In Column E we compute the distance from each point to the center, and then use an =IF function in Columns F:G to reproduce only those (x, y) values from Columns C:D whose distance from (a, b) is r , otherwise returning NA(). We insert the values in Columns F:G into the chart, showing the points as green markers. We can use a similar technique to generate other structures such as bisectors and analogs of various conic sections. With our model we vary the values of the center and radius to see the resulting output and to discover patterns. In using =IF statements involving equality with real numbers, we need to be aware of round-off considerations.

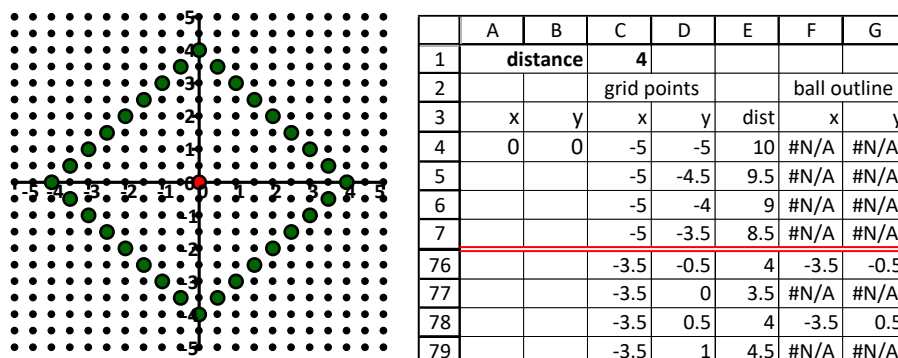
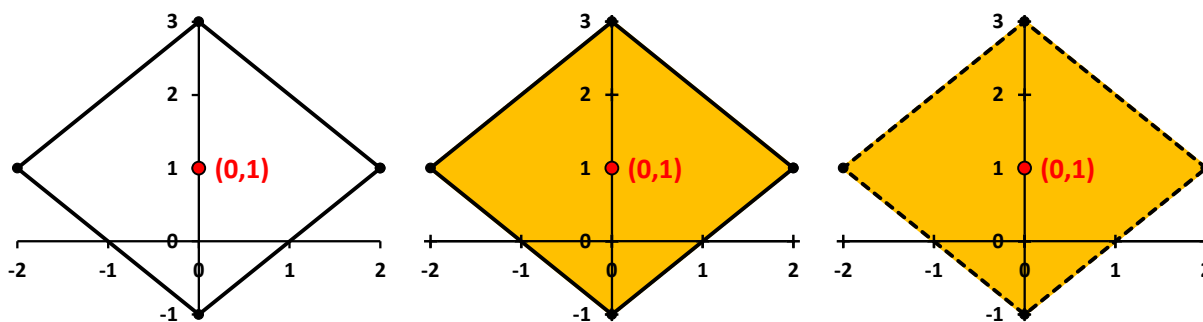


Figure 3.5. Taxicab Circle Special Layout

From the observations above, it is straight-forward to discover one way to create taxicab “circles” or balls of radius r . A closed ball contains the boundary, while the open one contains no boundary points. We simply plot the four points that are located r units horizontally and vertically from the center and connect them with line segments (Figure 3.6). We can also verify our findings algebraically if desired. We fill in the interior to produce a closed ball of radius 2 using a variation of the technique addressed previously.



Figures 3.6 Taxicab Circle with Corresponding Closed and Open Balls

Finding all points that are equidistant from two given points is more difficult to do in the taxicab metric. A good approach is first to examine the output a grid model. We see what may be surprising output in Figure 3.7a. By studying patterns and realizing that the location of the two points influences the resulting outcome, we can produce an animated version as seen in Figures 3.7b, although alternate constructions are possible.

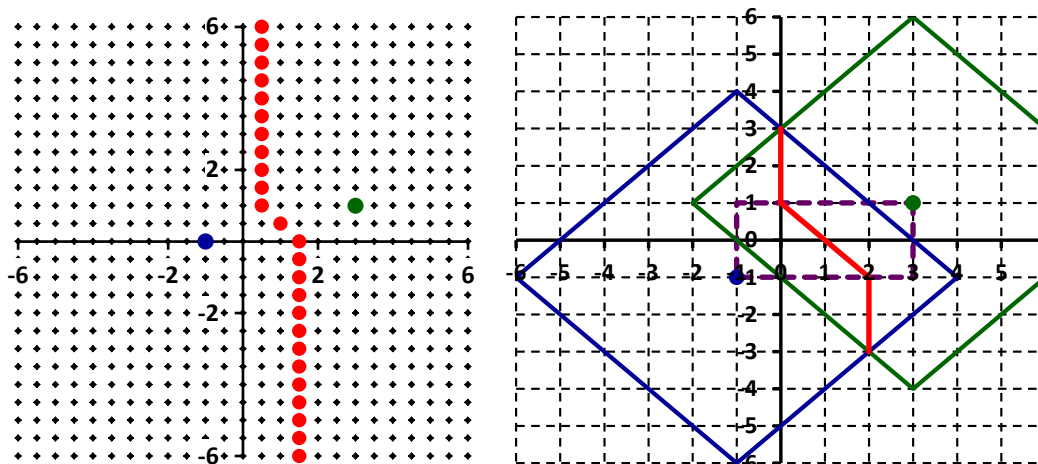


Figure 3.7 Locus of Points Equidistant from Two Given Points (Taxicab)

From our grid model we will find a bigger surprise. Sometimes, there is a whole region of points that are equidistant from the two points, as in Figure 3.8a. After further investigations, we can discover that this happens when the horizontal and vertical distances between the points are equal. We can then use our discoveries to create a more sophisticated animated graph such as in Figure 3.8b. This example is enhanced by using a scroll bar to vary r . Consult [5] to find a wealth of other interesting taxicab aspects to pursue.

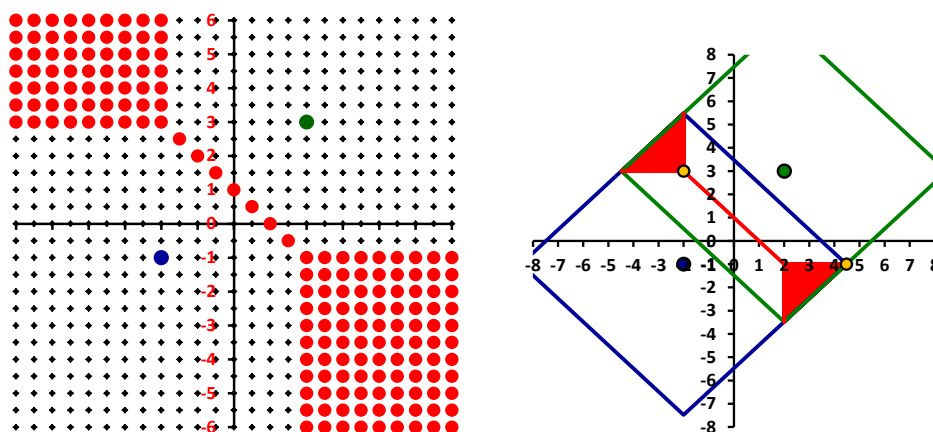


Figure 3.8 Special Case Equidistant Points

The taxicab metric provides students with a superb metric for creating a new class of geometric figures by adapting classical Euclidean definitions. For example, consider an ellipse as the set of all points for which the sum of the distances from two fixed points is a constant. We can explore *Excel* grid displays to discover the pattern for generating these as illustrated in Figure 3.9. We create grid points down Columns A:B and show the two fixed points in Columns C:D. We compute the distances from each grid point to fixed points in Columns E:F, and their sum in Column G. Then in Columns H:I we use =IF formulas to reproduce only those coordinates (x,y) the sum of whose

distances to the fixed points are a given value. By experimenting, we can see how to generate the taxicab ellipse from the corners of the rectangle determine by the fixed points.

	A	B	C	D	E	F	G	H	I
6	grid p		points					ellipse	
7	x	y	x	y	d(p,a)	d(p,b)	sum	x	y
8	-8	-8	4	4	24	15	39	#N/A	#N/A
9	-8	-7.5	0	-1	23.5	14.5	38	#N/A	#N/A
10	-8	-7			23	14	37	#N/A	#N/A
11	-8	-6.5			22.5	13.5	36	#N/A	#N/A
384	-2.5	-1.5			12	3	15	#N/A	#N/A
385	-2.5	-1			11.5	2.5	14	-2.5	-1
386	-2.5	-0.5			11	3	14	-2.5	-0.5
387	-2.5	0			10.5	3.5	14	-2.5	0

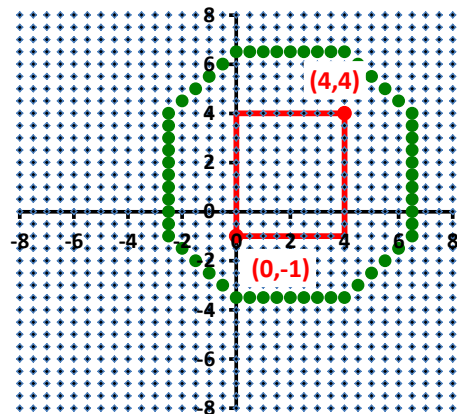


Figure 3.9 Discovering the Creation of Taxicab Ellipses

As a result of our discoveries we next use our observations to create an xy -graph as in Figure 3.10, where we also have used some creativity to show the taxicab distances from points on the ellipse. We then can animate our model to see how the distances vary at points around the ellipse by incorporating a scroll bar. We also can examine how the shape changes by varying the given distance.

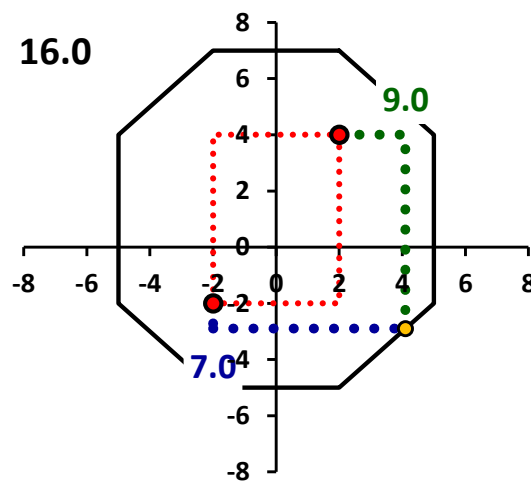


Figure 3.10 xy -chart of a Taxicab Ellipse

3.3. Maximal Metric: In the metric space for the maximal metric we again use $X = \mathbb{R}^2$, but the distance now is given by the expression $d((x_1, y_1), (x_2, y_2)) = \max(|x_2 - x_1|, |y_2 - y_1|)$. We can define a similar metric on \mathbb{R}^n as follows. Let $a = (x_1, x_2, \dots, x_n)$, $b = (y_1, y_2, \dots, y_n)$, then $d(a, b) = \max_{i=1, \dots, n} |x_i - y_i|$. We will notice (Figure 3.11) that balls in \mathbb{R}^2 are squares with sides parallel to the axes, while in \mathbb{R}^3 they are cubes, and in \mathbb{R}^n they are hyper-cubes.

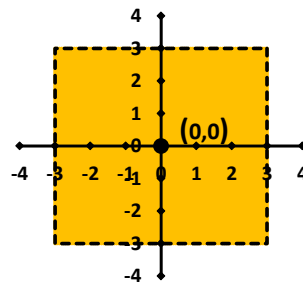


Figure 3.11 Open Ball in Maximal Metric

3.4. l_p Metrics: With $X = \mathbb{R}^2$, for any real number $p \geq 1$, we define the l_p metric by $d((x_1, y_1), (x_2, y_2)) = (|x_1 - x_2|^p + |y_1 - y_2|^p)^{1/p}$. If $p = 2$, then we get the Euclidean metric, while with $p = 1$ we have the taxicab metric. We can define a similar l_p^n metric in \mathbb{R}^n by $d((x_1, x_2, \dots, x_n), (y_1, y_2, \dots, y_n)) = (\sum_{i=1}^n |x_i - y_i|^p)^{1/p}$. As we see in Figure 3.12, with *Excel* we can create the boundary of a ball in \mathbb{R}^2 centered at (x_0, y_0) with radius r much as we did for the Euclidean metric, via the parametric equations

$$x = x_0 + r|\cos t|^{2/p}\text{sign}(\cos t), \quad y = y_0 + r|\sin t|^{2/p}\text{sign}(\sin t),$$

If we link the cell for the parameter p to a scroll bar in our model, then as we increase p we see what are sometimes called “super ellipses”. One can prove that for $p \geq 1$ our definition satisfies the requirements for a metric. However, for $0 < p < 1$ this is not the case, since the triangle inequality fails (Figure 3.12b). Nonetheless, the curves produced for $0 < p < 1$ have geometric significance. For example, the curve for $p = 2/3$ is an astroid [2]. Figure 3.12c shows unit circles for $p = 1, 2, 4, 1000$. As $p \rightarrow \infty$, the resulting metric approaches the l_∞ metric, which is the same as the maximal metric. Members of the l_p family of metrics are used in the fields of real analysis and statistics.

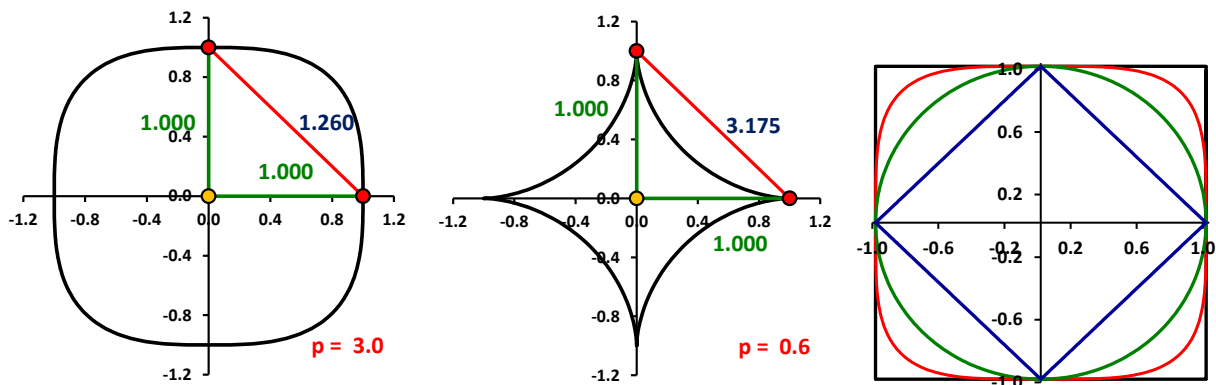


Figure 3.12 Unit Circles in l_p

3.5. British Rail Metric: For this metric on $X = \mathbb{R}^2$, the distance between two points $a = (x_1, y_1)$ and $b = (x_2, y_2)$ is defined by using the Euclidean distance that is needed to go first from a to the origin and then from the origin to b . Thus, $d(a,a) = 0$, while for $a \neq b$,

$$d(a, b) = (x_1^2 + y_1^2)^{1/2} + (x_2^2 + y_2^2)^{1/2}$$

This metric's name is based on the British rail system, where rail line routes usually require traveling via London. In fact, most internal travel in Papua New Guinea on Air Niugini similarly requires travel via Port Moresby. Thus, the Air Niugini distance from Madang to Goroka is $d(\text{MAG}, \text{GKA}) = d(\text{MAG}, \text{POM}) + d(\text{POM}, \text{GKA}) = 498 \text{ km.} + 426 \text{ km.} = 924 \text{ km.}$, while the actual direct distance is only 107 km. In Figure 3.13, both the airline's route map and its bird of paradise logo in were created in *Excel*, the latter using Euclidean circles almost exclusively.

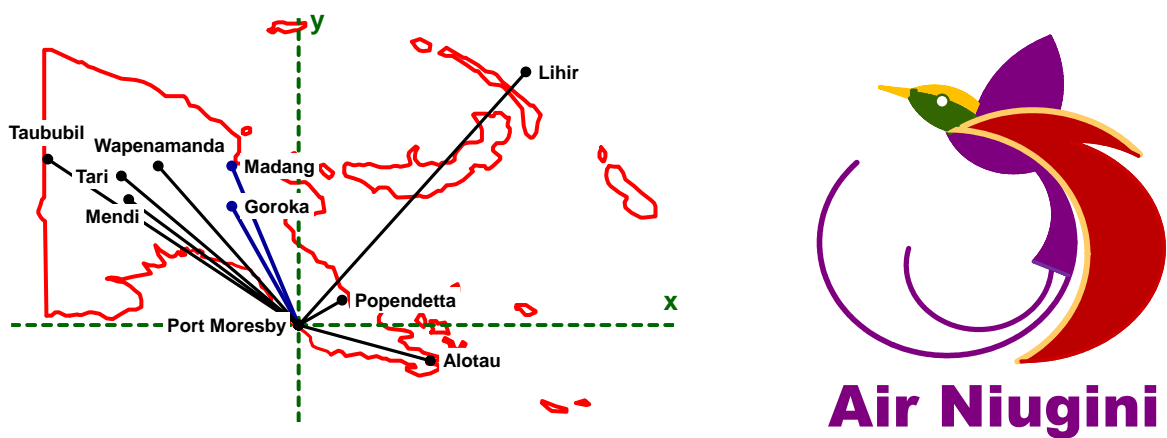


Figure 3.13 Air Niugini Flight Routes

3.6. Two Additional Interesting Metrics: Here we list two more metrics that can be studied by students. In the first, we let the X be the set of all finite closed intervals of the real line \mathbb{R} . We define the distance between intervals $A = [i, j]$ and $B = [m, n]$ by $d(A, B) = \max(|i-m|, |j-n|)$. Figure 3.14 illustrates the graphical output of an animated *Excel* model for this metric.

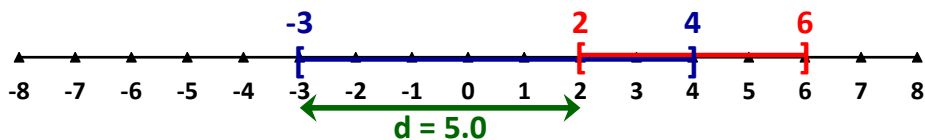


Figure 3.14 Distance between Closed Intervals

If d is any metric on a metric space X , then we can define a new metric, d' , by

$$d'(x, y) = \frac{d(x, y)}{1 + d(x, y)}$$

Starting with d as the usual Euclidean distance on \mathbb{R} , the new distance of any point from the origin is $-1 < d'(x, y) < 1$. In Figure 3.15 we use *Excel* to illustrate the distances from the origin to points on the real line as well as the triangle inequality.

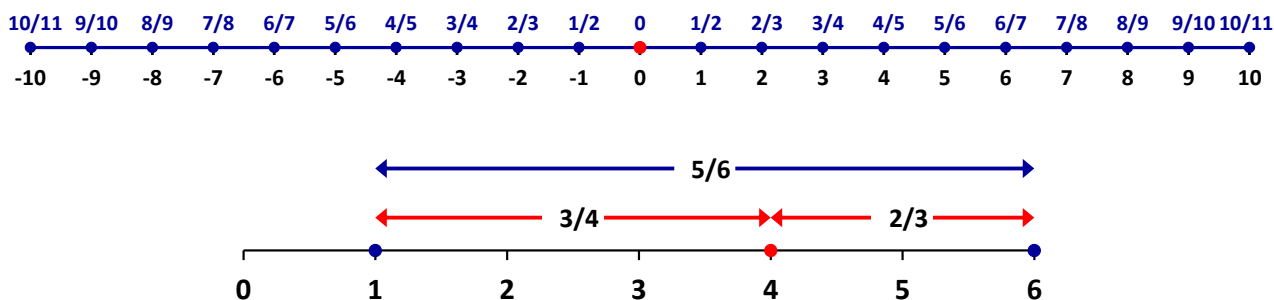


Figure 3.15 New Metric on \mathbb{R}

3.7. Great Circle Metric: Here we let X be the points in \mathbb{R}^3 on a sphere of radius ρ . The distance between points $a = (x_1, y_1, z_1)$ and $b = (x_2, y_2, z_2)$ is given by the shorter arc length of the great circle that passes through those points and the origin $(0,0,0)$. Using the dot product and related results from calculus or linear algebra, the angle (in radians) between the radii on the great circle to those points is given by $\theta = \cos^{-1}(\vec{a} \cdot \vec{b}) / (\|\vec{a}\| \|\vec{b}\|)$, where $\|\vec{v}\|$ denotes the length of a vector and \cdot is the dot product. Thus, $d(a, b) = \rho\theta$. *Excel*'s =SUMPRODUCT function gives the dot product.

We have designed our illustrations in Figure 3.16 as rough static displays. The first one shows distances for coordinates with rectangular coordinates, while the second one uses longitude and latitude. From calculus, the relationship between spherical and rectangular coordinate systems is given by $x = \rho \sin \varphi \cos \theta$, $y = \rho \sin \varphi \sin \theta$, $z = \rho \cos \varphi$, $\rho^2 = x^2 + y^2 + z^2$. Readers are challenged to create a fully-animated graphical display for this model, perhaps incorporating drawings of geographical maps on the sphere's surface.

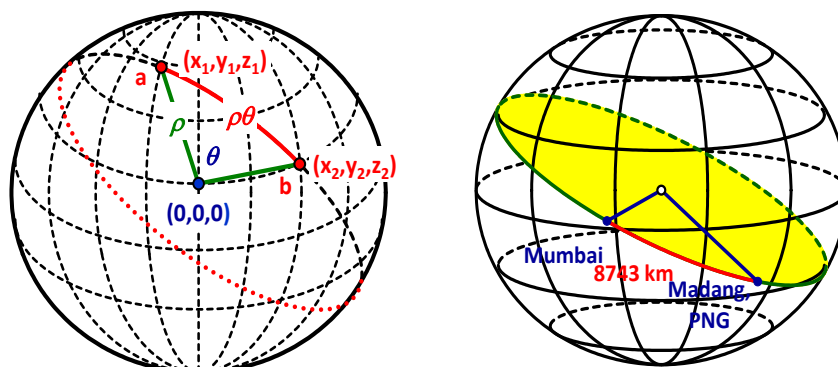


Figure 3.16 Spherical Distances

3.8. Continuous Real Functions – Distance Metric: Let $X = C[a, b]$ be the set of real-valued continuous functions on the interval $a \leq x \leq b$ with metric $d(f_1, f_2) = \max_{x \in [a, b]} |f_2(x) - f_1(x)|$.

With this definition of distance, it can be shown that X is a metric space. The definition says that the distance between two functions is determined at the point, x_0 , where the two functions are the farthest apart. This is illustrated by the graph in Figure 3.17b. We use the model of Figure 3.17a for functions in $C[0,1]$ to find the distance between $f_1(x) = x$ and $f_2(x) = x^n$, where n is a parameter (Cell C3) of the model.

There are at least three ways in which we can either determine or closely estimate this distance between two functions: (a) We can use calculus to compute the derivative of $f_1 - f_2$ to determine the maximum value of $|f_1(x) - f_2(x)|$. We enter the calculated value of x_0 in Cell E3, computing the distance between the functions in Cell E4. (b) We can input an estimate for the point x_0 at which the maximum distance occurs (Cell E3), enter the distance formula in Cell E4, and then use *Excel's* solver tool to find the maximum of E4 by varying E3. (c) We can use Column D to compute $|f_1(x) - f_2(x)|$ for each x , and use *Excel's* =MAX function to find the greatest in Column D. Column E lets us find the corresponding point, x_0 . Of course, (c) only gives an approximation, since the actual x_0 , as the maximum will seldom be exactly one of the values in Column D.

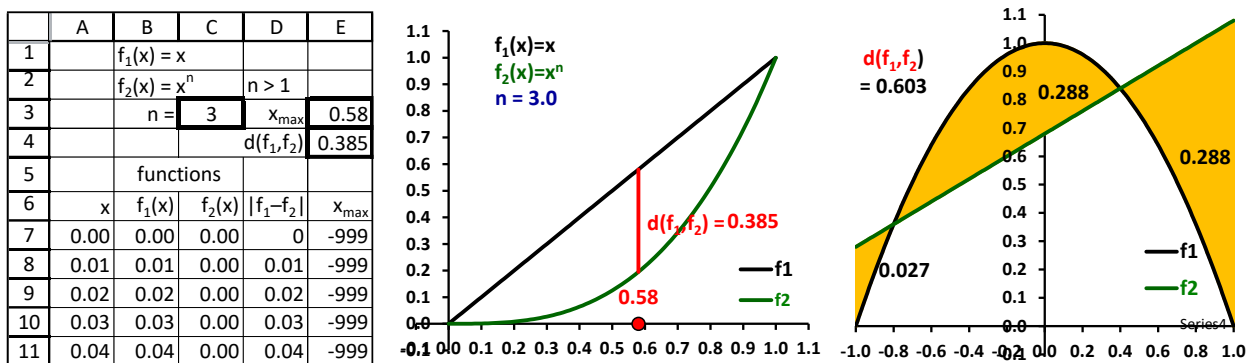


Figure 3.17 Metrics using the Maximum Distance and the Area between Functions

3.9. Continuous Real Functions – Area Metric: Let $x = C[a, b]$ be the space of continuous functions over the interval $a \leq x \leq b$, and $d(f_1, f_2) = \int_a^b |f_2(x) - f_1(x)| dx$. Thus, the distance between functions f_1 and f_2 is the net area between their graphs. We find this using calculus, although we could enter formulas for the areas of certain functions in advance. In the illustration of Figure 3.17c we have $f_1(x) = 1 - x^2$ and $f_2(x) = 0.4x - 0.68$ as functions in $C[-1,1]$. Because the curves intersect, we need to compute the integral in segments. We factor $f_2(x) - f_1(x) = x^2 + 0.4x - 0.32 = (x + 0.8)(x - 0.4)$, and evaluate the following integrals to find the area.

$$d(f_1, f_2) = \int_{-1.0}^{-0.8} (f_2(x) - f_1(x)) dx + \int_{-0.8}^{0.4} (f_1(x) - f_2(x)) dx + \int_{0.4}^{1.0} (f_2(x) - f_1(x)) dx \approx 0.60$$

4. Selected Topics from the Study of Metric Spaces

Besides using *Excel* to illustrate and find distances for metric spaces, we can use its models to discover patterns, illustrate topology and geometry properties, and understand the ideas behind proofs of such topics as the continuity of functions, the convergence of sequences, and others through creative spreadsheet techniques.

4.1. Open and Closed Sets: A subset S of a metric space X is open if for each $x_0 \in S$ there is a real $\varepsilon > 0$ for which the open ball $S(x_0, \varepsilon)$ centered at x_0 is contained in S . In Euclidean \mathbb{R} space, using scroll bars to vary x_0 and ε , in Figure 4.1 we illustrate the definition, by showing that the open interval $S = (0,1)$ is an open set, while $(0, 1]$ is not. Thus, we see that for $x_0 = 0.8 \in S = (0,1)$, with $\varepsilon = 0.1$ the interval $(x_0 - \varepsilon, x__0 + \varepsilon) \subseteq S$, with a similar result holding for any $x_0 \in S$. On the other hand, we notice that if $x_0 = 1.0 \in S' = (0,1]$, then no matter how small we choose ε , the interval $(x_0 - \varepsilon, x_0 + \varepsilon)$ contains points that are not in S .

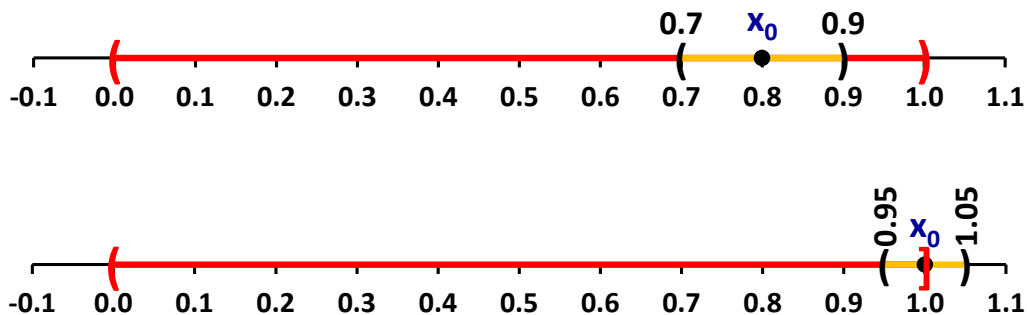


Figure 4.1 Open and Non-open Sets in \mathbb{R}

The diagrams in Figure 4.2 come from a similar model for \mathbb{R}^2 in which we can vary the metric used in the illustration by clicking a spin button, and vary the radius of an open ball with a scroll bar. We see that the physical size of these balls of the same radius vary with the metric that is used.

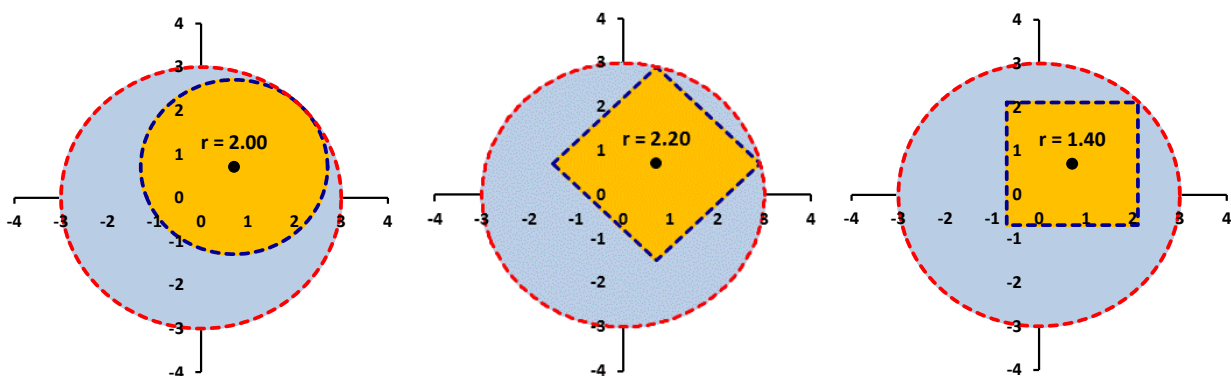


Figure 4.2 Inserting an Open Ball in an Open Set for Different Metrics

4.2. Visualizing Proofs and Definitions: In a class that involves proving results, many students initially have difficulties in learning how to carry out, or even to understand, proofs. In a standard classroom, instructors typically help students to visualize these aspects by drawing images on a board. Although it takes more time to create similar images on *Excel*, doing so can allow us to create animated drawings that can provide additional insights into these aspects.

In Figure 4.3a we see a drawing illustrating a step in the proof that the intersection of three open sets is open. Figure 4.3b illustrates how to produce an open ball centered at a given point so that it fits inside another open ball. Thus if the larger open ball centered at (x_0, y_0) has radius r_0 and a point (x_1, y_1) is d units away from the center, then an open ball centered at (x_1, y_1) with radius $r_0 - d$ is contained within the larger ball. Both students and their instructors can create similar 2-dimensional models for a variety of theorems and illustrated by various metrics.

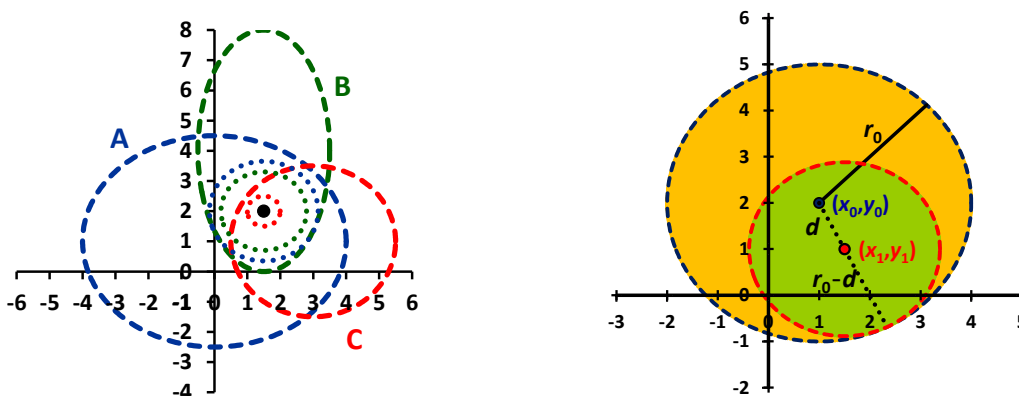


Figure 4.3 Illustrating Proofs Concerning Open Sets

In Figure 4.4 we illustrate the definition of the diameter of an arbitrary set (here using the Euclidean metric in \mathbb{R}^2), by choosing two arbitrary points on the boundary of a set created by the polar equation $r = \cos^3 x + \sin^3 x$, and then using *Excel's* solver to maximize the distance between two points, while including constraints that insure points lie on the boundary.

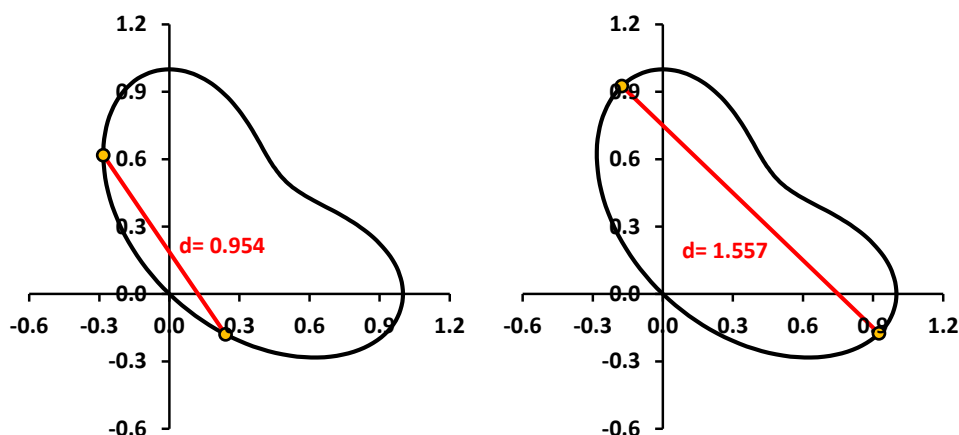


Figure 4.4 Visualizing the Diameter of a Set (before and after Solver)

4.3. Limits, Continuity, and Convergence: In this section we look at a brief selection of basic topics from metric spaces, calculus, and similar topics.

We say that a sequence of points p_1, p_2, p_3, \dots of a metric space X converges to a point a if for any $\varepsilon > 0$ there is a positive integer N so that if $n \geq N$, then p_n is in the open ball of radius ε centered at a . In Figure 4.5 we see a sequences of points (x_n, y_n) in \mathbb{R}^2 that converges to $(1.0, 2.0)$. If $\varepsilon = 0.83$, then all of the points p_n for $n \geq 8$ are contained in the open ball of radius ε centered at a (although some earlier elements are also in the ball), while if $\varepsilon = 0.45$, then all of the points p_n lie in the required open ball if $n \geq 20$, but p_{19} does not. Our model can be animated by attaching the cell containing the value of ε to a scroll bar.

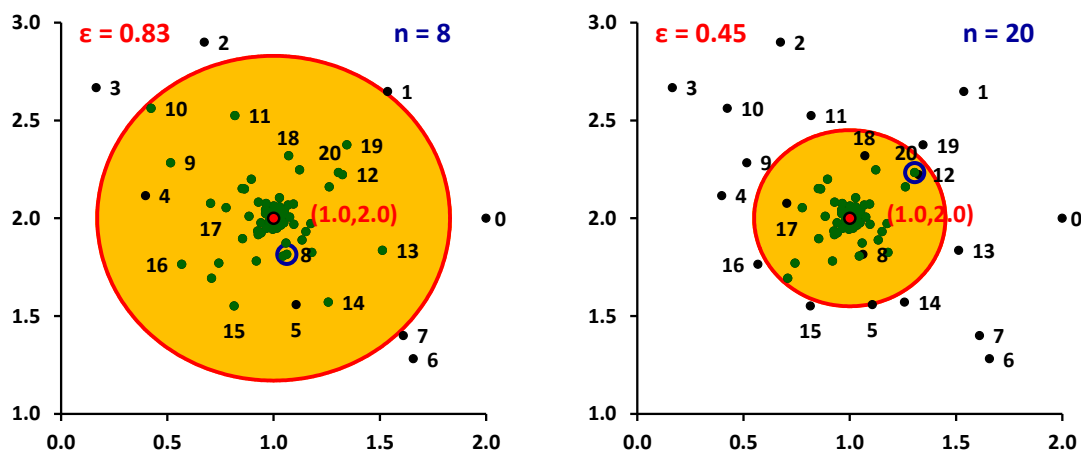


Figure 4.5 Convergence of a Sequence

One of the basic goals in an analysis course is to prove results involving limits and continuity of functions. In Figure 4.6 we see the output of an interactive model for illustrating that the function $f: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ given by $f(x, y) = (x^2 - y^2, xy)$ is continuous at the point $a = (0.65, 0.50)$.

In Figure 4.6a, we generate grid points throughout the square region, $-1 \leq x \leq 1, -1 \leq y \leq 1$, and produce their images in Figure 4.6b. Next, we plot the points a and $f(a)$ in the respective graphs, and for an $\varepsilon > 0$ produce a green ball of radius ε around $f(a)$ in the right graph. Next we choose a $\delta > 0$, form a red/yellow ball of radius δ around point a in the left graph, and then generate the red/yellow image of that set in the right graph. Finally, we attach a scroll bar to the cell that contains δ , and use it to adjust the value of δ until the image set is contained within the green set. We can also do the mathematical analysis to evaluate the required δ and modify the model so it updates automatically when δ is changed.

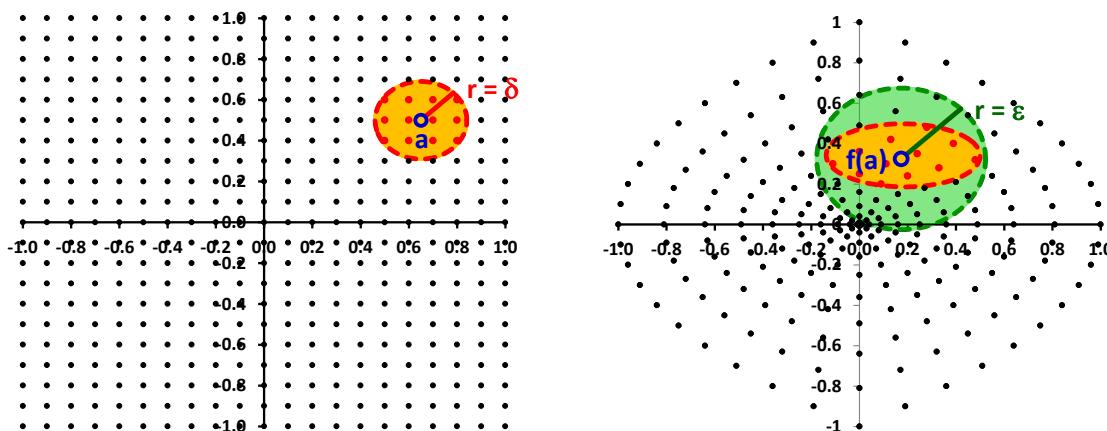


Figure 4.6 Interactive Illustration the Concepts of a δ - ε Proof in \mathbb{R}^2

We next display two versions of traditional ways to visualize a δ - ε proof of the continuity of a function $f: \mathbb{R} \rightarrow \mathbb{R}$ at the point $x = 2$, where $f(x) = x^2$. In Figure 4.7, using the concept of limit, given an $\varepsilon > 0$, we see a blue open interval of radius ε about $f(a)$. We then select a $\delta > 0$ to produce a red open interval about a , using a scroll bar to vary δ so that the image of the red interval lies within the blue ε -interval. We can also modify our model by calculating δ as a function of ε so that the graph is updated automatically as ε is varied.

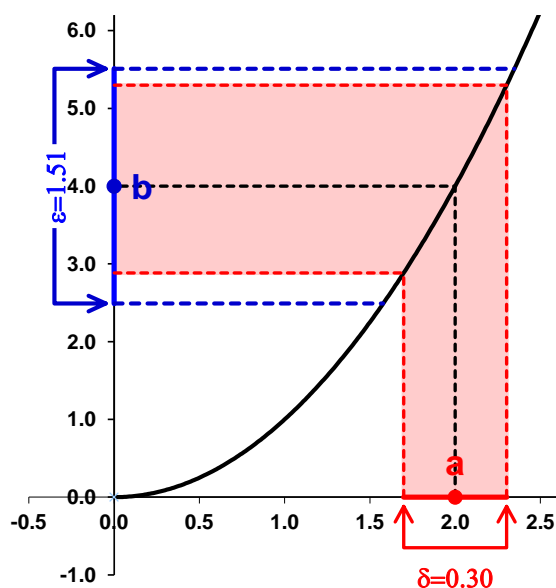


Figure 4.7 Interactive Illustration the Concepts of a Traditional δ - ϵ Proof in \mathbb{R}

In Figure 4.8 a second version of the previous example consists of two intervals, one for the domain and the other for the range. It is a one-dimensional version of the approach that we used in our earlier two-dimensional model. Again we vary the ϵ - and δ -intervals with scroll bars.

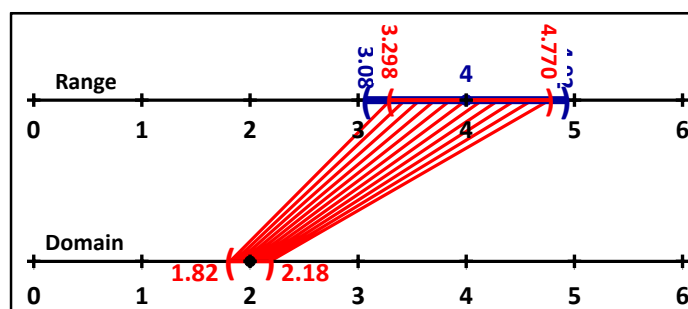


Figure 4.8 Alternate Illustration of a δ - ϵ Proof in \mathbb{R}

5. Classical Inequalities

By its nature, the study of metric spaces involves using a variety of classical inequalities, especially in verifying the triangle inequality for distances. There are a number of resources available to both the proofs and visualization of inequalities [1], [7], [8]. Here we present two.

First, in Figure 5.1 we show an animated geometric *Excel* model animated using scroll-bars for a two point version in \mathbb{R}^2 for the classical Cauchy-Schwarz inequality. Also see [10].

Since the white areas are equal, and the area of a trapezoid is less than or equal to a rectangle with the same sides, we see that for $x_i \geq 0, y_i \geq 0$

$$x_1y_1 + x_2y_2 = A_{trap} \leq \sqrt{x_1^2 + x_2^2}\sqrt{y_1^2 + y_2^2}, \text{ or } (\sum x_i y_i)^2 \leq \sum x_i^2 \sum y_i^2$$

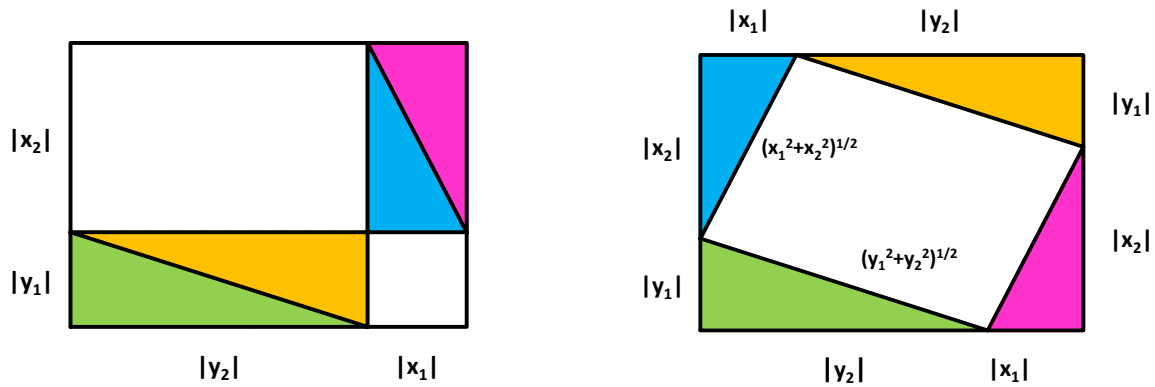


Figure 5.1 Cauchy-Schwarz Inequality in \mathbb{R}^2

Next we look at Hölder's Inequality. As a preliminary step, we illustrate Young's Inequality [9]: Let f and g be strictly increasing functions on $[0, \infty)$, with $g = f^{-1}$, and $f(0) = g(0) = 0$. Then for $a, b \geq 0$,

$$ab \leq \int_0^a f(x) dx + \int_0^b g(y) dy$$

In the Figure 5.2, the curve represents $y = f(x)$ relative to the x -axis, and $x = f^{-1}(y)$ relative to the y -axis. The yellow area gives the first integral, and the purple area is the second integral. In both cases shown, the area of the rectangle, ab , does not exceed the sum of the areas under the curves. We use the functions $f(x) = x^2$ and $g(y) = y^{1/2}$ in the illustration.

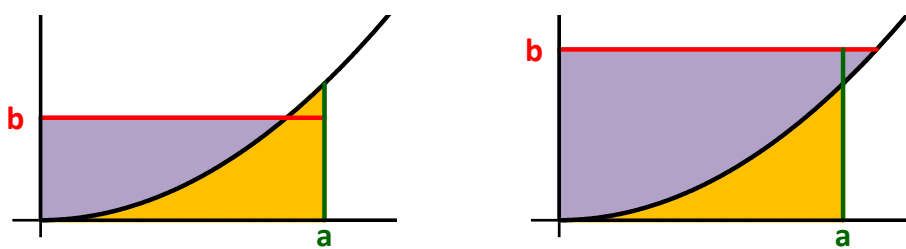


Figure 5.2 Young's Inequality

Then, if $p > 1$, let $f(x) = x^{p-1}$ and $g(y) = y^{1/(p-1)}$. Next, using Young's inequality with $q = p/(p-1)$, we have $(1/p) + (1/q) = 1$ and

$$ab \leq \int_0^a x^{p-1} dx + \int_0^b y^{1/p-1} dy = \frac{a^p}{p} + \frac{p-1}{p} b^{p/(p-1)} = \frac{a^p}{p} + \frac{b^q}{q}$$

with equality when $a^p = b^q$.

Now let

$$A = \left(\sum_{i=1}^n |a_i|^p \right)^{1/p} \quad \text{and} \quad B = \left(\sum_{i=1}^n |b_i|^q \right)^{1/q}$$

With $a = |a_i|/A$, $b = |b_i|/B$ and $(1/p) + (1/q) = 1$, $p > 1$, $q > 1$,

$$\frac{|a_i|}{A} \frac{|b_i|}{B} \leq \frac{(|a_i|/A)^p}{p} + \frac{(|b_i|/B)^q}{q}$$

so that summing these terms

$$\frac{1}{AB} \sum_{i=1}^n |a_i b_i| \leq \frac{1}{pA^p} \sum_{i=1}^n |a_i|^p + \frac{1}{qB^q} \sum_{i=1}^n |b_i|^q = \frac{1}{p} + \frac{1}{q} = 1$$

Multiplying by AB then gives Hölder's inequality

$$\sum_{i=1}^n |a_i b_i| \leq \left(\sum_{i=1}^n |a_i|^p \right)^{1/p} \left(\sum_{i=1}^n |b_i|^q \right)^{1/q}$$

6. Some Related Projects

There is a large range of other metrics to examine [4]. For projects, readers can consult the references for both metric spaces and other topics. Also, books on operations research, calculus, and geometry can provide other distance-related topics. We indicate some of these examples below.

Figure 6.1 illustrates an algorithm to find the minimum distance from an initial node, S , to additional nodes in a network. One can design an interactive diagram, either with user-entered selections at each stage of an algorithm, or completely computed by the spreadsheet. The field of operations research provides us with a large selection of such applications. Figure 6.2 shows animated *Excel* classical physical Euclidean constructions.

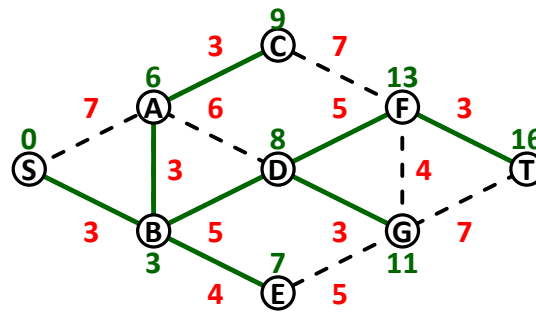


Figure 6.1 Minimal Distance Network Algorithm

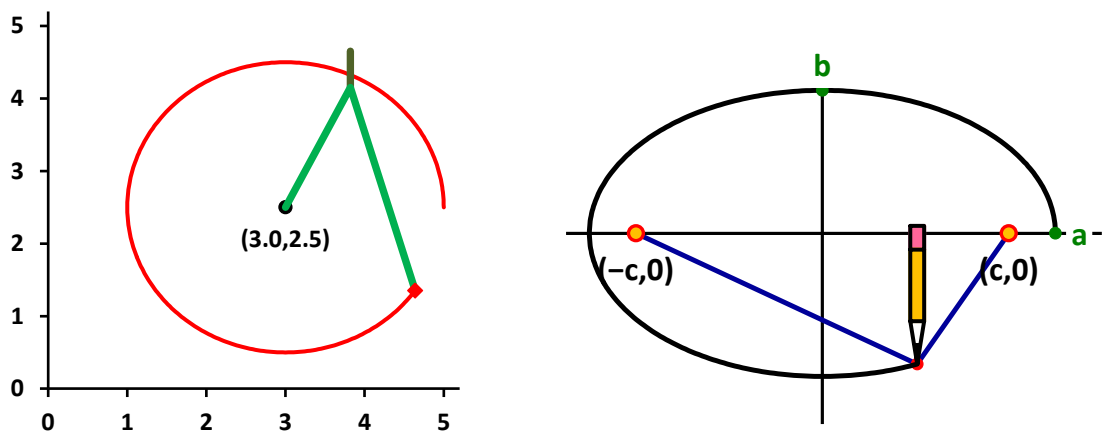


Figure 6.2 Two Geometric Constructions

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- [9] Tolsted, E. (1964). “An Elementary Derivation of the Cauchy, Hölder, and Minkowski Inequalities from Young’s Equation”. *Mathematics Magazine*, vol. 37. Washington, DC: Math. Assn. of Am.

Supplementary Electronic Materials

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- [11] Arganbright, D. [1 Road Distance](#), *Excel 2010* file.
- [12] Arganbright, D. [2 Drawing Circles](#), *Excel 2010* file.
- [13] Arganbright, D. [3 Taxicab Metric](#), *Excel 2010* file.
- [14] Arganbright, D. [4 Euclidean Bisecting](#), *Excel 2010* file.
- [15] Arganbright, D. [5 Taxicab Bisecting](#), *Excel 2010* file.
- [16] Arganbright, D. [6 Taxicab Ellipses](#), *Excel 2010* file.
- [17] Arganbright, D. [7 Metric Space Balls](#), *Excel 2010* file.
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